Casting Out Nines

The “digital root” of a number is the result you get if you add up its digits, and then add up the digits of that result, and so on, until you end up with a single digit. For instance, the digital root of 44689 is computed by finding that \(4 + 4 + 6 + 8 + 9 = 31\), and then \(3 + 1 = 4\) gives you a single-digit answer.

1. Let's look at two numbers that add up to 44689, such as 31847 and 12842. What relationship can you find among the digital roots of these numbers?

2. What about two numbers that subtract to make 44689, like 83491 and 38802? Is there a relationship among their digital roots? What can you do with 100000 and 55331?

3. What about two numbers that multiply to make 44689, like 67 and 667? Or two other numbers that multiply to make 44689, like 23 and 1943?

4. The process of taking the digital root is called “Casting out nines” for a reason: what you're actually doing in computing the digital root is another way of determining the remainder when you divide by 9. In other words, you keep throwing away multiples of 9 until you're eventually left with a number smaller than 9. Well, that's not quite true: why not?

5. In the original example of 44689, we obtained 31 after the first step. Let's see the 9s disappearing as we go from 31 to 3 + 1: 31 means \(3 \times 10 + 1\) which is the same as \(3 \times 9 + 3 \times 1 + 1\), so after throwing away the 9s we have \(3 \times 1 + 1\), which finally is \(3 + 1\). Can you give a similar explanation for how 44689 turns into \(4 + 4 + 6 + 8 + 9\) after throwing away a lot of 9s?

6. One of the major uses of casting out nines is to check arithmetic quickly. If your calculation (like in the first few problems here) doesn't match up, then you know there was an arithmetic mistake. Which of the following can be proved wrong by casting out nines? Are the other ones actually correct?
   a) \(1234 + 5678 = 6812\)
   b) \(12345 – 9876 = 2469\)
   c) \(10101 – 2468 = 7623\)
   d) \(1234 \times 5678 = 7006652\)
   e) \(4321 \times 8765 = 37783565\)
   f) \(345 \times 543 = 196335\)
   g) \(2^7 = 130072\) (warning! How should you handle exponents? Think about this very carefully!)
7. On the other hand, certain kinds of mistakes will never be found by casting out nines. Can you give some examples of these? Examples that might be common?

8. Why is this process a bad idea for division when it works so well for addition, subtraction, and multiplication? Give an example where casting out nines seems to be “wrong” even though the answer is correct.

9. On the other hand, you can use casting out nines to check division problems by rewriting them as multiplication and addition. How would you rewrite “23894 divided by 82 is 291 with a remainder of 32” using only multiplication and addition, so you could then check it by casting out nines?

10. Another way to think about casting out nines is that as you add 9 to a number, you increase the tens digit by 1, and decrease the ones digit by 1, so adding 9 won't change the digital root. What is the flaw in this logic? Can you repair it?

11. Casting out nines has some other interesting applications as well. What is the digital root of 3726125? Can you use that information to explain why 3726125 is not a perfect square?

12. You can also cast out elevens instead of nines. Start with the rightmost digit, and alternately add and subtract. So with 44689 you'd take 9 – 8 + 6 – 4 + 4 = 7. If you end up with a negative number, remember you're casting out elevens, so just add 11 as many times as you'd like. Can you explain why this process works?

13. There are some common mistakes that you wouldn't be able to catch with casting out nines, but you can catch by using casting out elevens. Give at least one example.

14. There's a magic trick that is most often done using a calculator. Pass the calculator around the room, and each person types in one digit and presses the multiplication key. After a while, the calculator screen is full of digits. The person holding the calculator at that point eliminates any one digit 1 through 9 (not 0), and then takes the remaining digits and writes them in any order. For example, they might write 3004129. Then, a mathematician almost instantly says what the missing digit is. Which digit is missing? How could the mathematician know? But sometimes the mathematician is wrong. Why?

15. What is the digital root of 4444•4444•4444•4444? Can you determine how many times you will have to sum the digits before obtaining a single digit answer?