26th Bay Area Mathematical Olympiad

Problems and Solutions

Mar 5, 2025

The problems from BAMO-8 are A–E, and the problems from BAMO-12 are 1–5.

Problems

A Imagine a language that uses the Latin alphabet and in which all words consist of six letters. Let us define the distance between any two words in that language as follows. If the letters in the two words are $a_1, a_2, \ldots, a_6$ and $b_1, b_2, \ldots, b_6$, then the distance between them is the number of values of index i, for $i = 1, \ldots, 6$, such that $a_i \neq b_i$. For example, the distance between these two words:

```
zmootq
zoeutq
```

is 3, because their letters differ in three positions — the 2nd, 3rd and the 4th.

(a) Find three words such that the distance between any pair of these words equals 2.

(b) Find three words such that the distances between the three pairs of these words are 5, 4 and 3.

The words do not have to make sense in English.

B Jessica has five distinct integers. She can form ten distinct pairs of these integers if she ignores the order within each pair. The sums of all ten pairs are known and they are:

$$-17, -14, -9, -6, -1, 2, 7, 12, 15, 23.$$

What are the five integers?

C/1 Fifteen balls labeled $1, 2, \ldots, 15$ are stored in two boxes, so that each box contains at least two balls and the average of the numbers on the balls in one box is not equal to the average of the numbers on the balls in the other box.

Show that it is possible to move one ball from one box to the other so that either both boxes' averages increase or both boxes' averages decrease.

D/2 Let S be a finite, nonempty set of points in the plane such that, for every point $A \in S$, there exist points $B, C \in S$ (distinct from A) such that $\angle BAC = 125^\circ$. What is the smallest possible number of points in S?

E/3 A *string* is a finite ordered sequence of one or more letters. For example, *apple* and *qvdkxo* are strings consisting of English letters. If one string appears as a consecutive sequence of letters inside another string, we say the first string is a *substring* of the second. (A string is considered to be a substring of itself.)

A *palindrome* is a string that remains the same regardless of whether its letters are read from left to right or from right to left. Given a string, we may count its distinct palindrome substrings. For example, the string *propeller* contains 7 distinct palindromes as substrings: *p*, *r*, *o*, *e*, *l*, *ll*, and *elle*.

What is the maximum number of distinct palindromes that can appear as substrings in a string of length 2025?

4 Dana wants to draw a circle (C_0) and six circles ($C_1, C_2, \dots, C_6$) around it in such a way that:

- C_0 is externally tangent to C_i for $i = 1, 2, \dots, 6$;
- C_i is externally tangent to C_{i+1} for $i = 1, \dots, 5$ and C_6 is externally tangent to C_1;
- the radius of C_i is a^{n_i}, for $i = 0, 1, \dots, 6$, where $a > 1$ and the n_i are distinct non-negative integers.

Show that Dana can succeed by exhibiting a construction that works, or provide a proof that no such construction exists.

5 The integers $1, 2, 3, \dots, 101$ are written on a blackboard. Sam and Nico take turns crossing out numbers one by one, starting with Sam, until three numbers a, b, c remain. Prove that Sam can guarantee

$$\frac{ab + bc + ca}{a^2 + b^2 + c^2} < \frac{8}{9}.$$

Solutions

A Solution:

(a) To make 3 words, W_1, W_2 and W_3, equidistant from one another with the distance of 2, it's sufficient to make 3 strings with identical characters, except for two arbitrarily chosen positions that are different in all 3 strings. Here is an example:

$W_1 =$ `search`
$W_2 =$ `snatch`
$W_3 =$ `stanch`

These three words differ in the 2nd and 4th positions only. Needless to say, the answer is not unique, which applies to question (b) too.

(b) Let us start with two words, W_4 and W_5, that are 5 units apart. Here is an example:

$W_4 =$ `abcdef`
$W_5 =$ `aghijk`

We will keep these W_4 and W_5 intact and come up with W_6 that is 4 and 3 units apart from W_4 and W_5, respectively. In order to meet the former requirement, first we should create W_6 that differs from W_4 in 4 positions. This example will work:

$W_4 =$ `abcdef`
$W_6 =$ `ablmno`

However, this W_6 is 5, rather than 3, units away from W_5. Now we need to bring W_6 closer to W_5 by 2 units, without affecting the distance between W_4 and W_6. That can be accomplished by changing the last 2 letters in W_6 above from `no` to `jk`. The final answer is:

$$W_4 = \texttt{abcdef}$$
$$W_5 = \texttt{aghijk}$$
$$W_6 = \texttt{ablmjk}$$

B Solution: Let the integers be $a < b < c < d < e$. Note that the smallest and largest pairwise sums must be $a+b$ and $d+e$; also, the second-smallest and second-largest pairwise sums must be $a+c$ and $c+e$.

If we write the ten equations explicitly:

$$\begin{aligned} a+b &= -17 \\ a+c &= -14 \\ &\vdots \\ d+e &= 23 \end{aligned}$$

and sum them up, the result is $4(a+b+c+d+e) = 12$. (Each letter is present in the left-hand sides four times, while the right-hand sides are simply all pairwise sums, regardless of their order.) Therefore,

$$a+b+c+d+e = 3.$$

Since

$$a+b+d+e = -17+23 = 6,$$

we must have $c = -3$. Since we know the values of $a+c$ and $c+e$, now we can deduce $a = -11$ and $e = 18$. Finally, since we know the values of $a+b$ and $d+e$, we conclude that $b = -6$ and $d = 5$. So the five numbers are -11, -6, -3, 5, and 18.

C/1 Solution: First, notice that the total average of the numbers on all 15 balls is $(1+15)/2 = 8$. This overall average must be somewhere between the average for the first box and the average for the second box, which we know are not equal. Thus, the average for one box is less than 8, and the average for the other box is greater than 8. Let us call these boxes A and B, respectively.

Now we consider two cases.

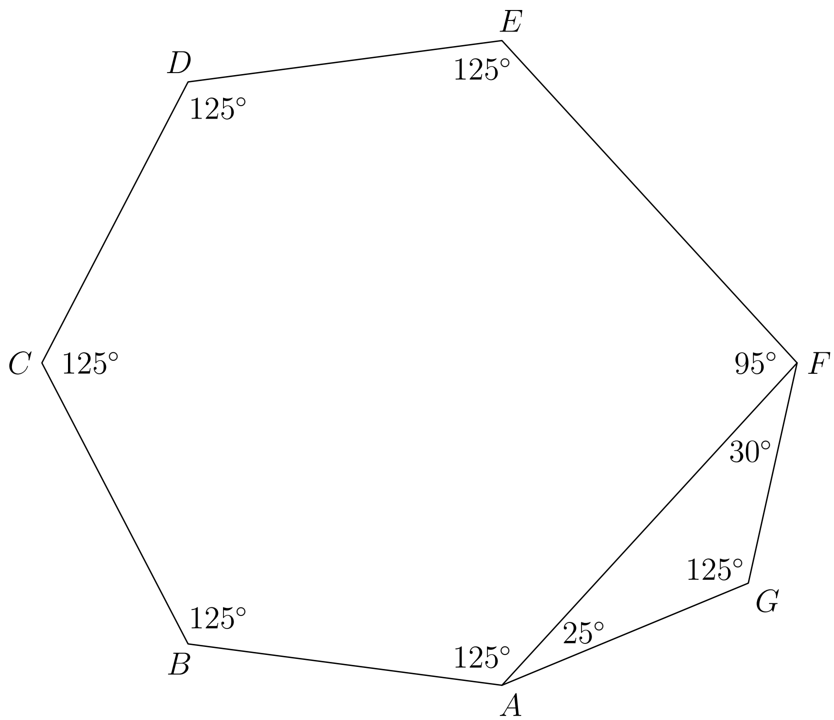
Case 1: The ball labeled 8 lies in box A. In this case, let us move this ball to box B. The average in box A decreases, since 8 was above average for box A. The average in box B also decreases, since 8 was below average for box B.

Case 2: The ball labeled 8 lies in box B. In this case, let us move this ball to box A. The average in box B increases, since 8 was below average for box B. The average in box A also increases, since 8 was above average for box A.

Therefore, in either case, moving ball 8 has the desired effect.

D/2 Solution: We will show that the smallest possible number of points in S is 7.

First, we show that such a set S with 7 points exists. Consider the heptagon drawn below:



It is very credible that a heptagon meeting these specifications exists (in fact there are many such heptagons), but for the sake of completeness, we describe how to construct one. Begin by arranging four segments of equal length end-to-end making 125° angles (all on the same side) at the joints, forming the path $ABCDE$. Then lines may be extended from A and E to create two more 125° angles at A and E, again on the same side of the path as the 125° angles at B, C, and D. These lines intersect at F, completing a hexagon $ABCDEF$. Since the sum of angles in a hexagon is 720°, we have $\angle EFA = 95^\circ$. Finally, we may erect a triangle $\triangle FGA$ on the outside of edge FA, whose angles are as shown in the diagram.

The set $S = \{A,B,C,D,E,F,G\}$ now meets the requirements set out in the problem, since

$$\angle FAB = \angle ABC = \angle BCD = \angle CDE = \angle DEF = \angle EFG = \angle FGA = 125^\circ.$$

Next, we show that any set S meeting the requirements contains at least 7 points.

Let S be such a set, and consider the convex hull of S, which is a convex polygon $\mathscr{P}$ whose vertices are some subset of S.

Let A be any vertex of $\mathscr{P}$. For any $B,C \in S$, angle $\angle BAC$ is contained in the interior angle of $\mathscr{P}$ at A. Since there exist $B,C \in S$ such that $\angle BAC = 125^\circ$, we conclude that each interior angle of $\mathscr{P}$ is at least 125°. Therefore, each exterior angle of $\mathscr{P}$ is at most 55°.

Since the exterior angles of a convex polygon add up to 360°, $\mathscr{P}$ must have at least $\lceil \frac{360}{55} \rceil = 7$ vertices. Thus $|S| \geq 7$.

E/3 Solution: We will show that if a string has n letters, then the maximum number of distinct palindromes occurring in it is simply n.

Examples of such strings are easy to find: for instance, one may take a string consisting of n different letters, or a string consisting of a single letter repeated n times (this second example works even if our alphabet has only one letter).

Now we will prove that a string cannot contain more than n distinct palindromes.

For each distinct palindrome occurring in a given string, underline the first occurrence of that palindrome. For example, in the string *propeller*, we would underline as follows:

`p r o p e l l e r`

Claim. Two underlined substrings cannot end at the same position.

We argue by contradiction. Suppose two underlined substrings end at the same position, as in this (deliberately incorrect) example:

`madamimadam`

In this case, the shorter palindrome occurs at the end of the longer palindrome. But that means the shorter palindrome *also* occurs at the *beginning* of the longer palindrome. Thus, the underlined copy is not the first occurrence of the shorter palindrome, and cannot have been underlined! In the example above, for instance, we should have underlined the first `madam`, not the second:

`madamimadam`

This completes the proof of the claim.

Since there are n positions where a substring can end, and no more than one underlined palindrome ends at each position, the number of distinct palindromes is at most n, or 2025 in our case.

4 Solution: Let $a > 1$ be fixed and arbitrary.

We claim that the following values for $n_0, n_1, \ldots, n_6$ will satisfy the conditions of the problem:

$$n_0 = 3 \qquad n_1 = 5 \qquad n_2 = 2 \qquad n_3 = 0 \qquad n_4 = 1 \qquad n_5 = 4 \qquad n_6 = 6$$

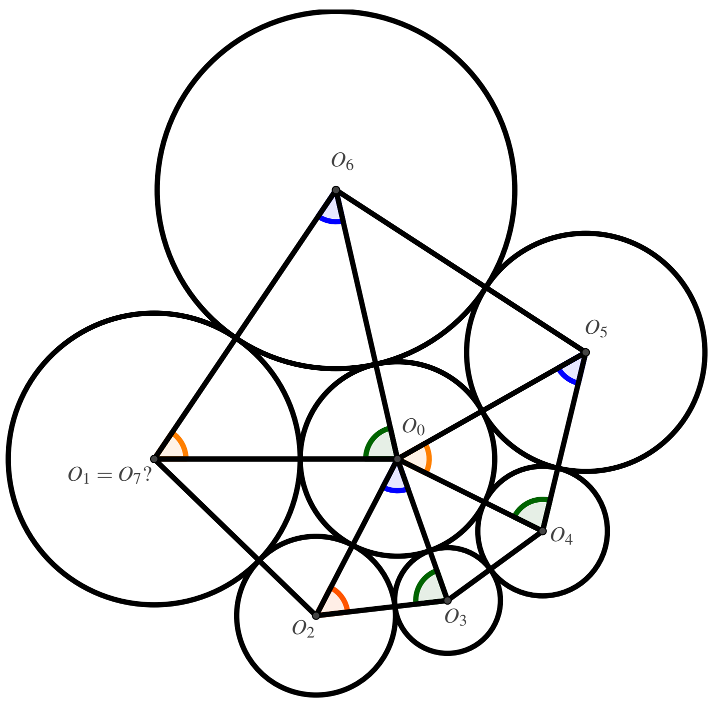
In other words, the radius of C_0 is a^3, the radius of C_1 is a^5, and so on.

Start with the center circle C_0 and construct circle C_1 tangent to it with radius a^{n_1}. Proceeding counterclockwise, construct circle C_2 tangent to C_1 and C_0, and then circle C_3 tangent to C_2 and C_1, etc. After constructing C_6 tangent to C_0 and C_5, construct an *eighth* circle, C_7, tangent to C_6 and C_0. Let C_7 have radius a^5, the same as C_1. We will be done if we can show that C_1 and C_7 coincide.

Let O_i denote the center of circle C_i (for $0 \le i \le 7$). The lengths of segments O_0O_6, O_6O_7, O_7O_0 are respectively $a^3 + a^6$, $a^5 + a^6$, $a^3 + a^5$. Likewise, lengths O_4O_5, O_5O_0, O_0O_4 equal $a + a^4$, $a^3 + a^4$, $a + a^3$, respectively. Finally, lengths O_3O_0, O_0O_2, O_2O_3 equal $1 + a^3$, $a^2 + a^3$, $1 + a^2$. We see that

$$\triangle O_0O_6O_7 \sim \triangle O_4O_5O_0 \sim \triangle O_3O_0O_2.$$

The corresponding angles are colored accordingly below.



Using similar reasoning, we see that

$$\triangle O_0O_1O_2 \sim \triangle O_4O_0O_3 \sim \triangle O_5O_6O_0.$$

We thus have two trios of similar triangles. The central angles $\angle O_iO_0O_{i+1}$, for $1 \le i \le 6$, constitute the complete sets of internal angles of two triangles; thus they add up to 360 degrees, placing O_1 and O_7 on the same ray emanating from O_0. Since C_1 and C_7 additionally have the same radius, we conclude that $C_1 = C_7$.

Remark. For values of a sufficiently close to 1, it is possible to extend the arrangement of circles in this problem to a circle packing covering the whole plane, so that every circle is surrounded by a ring of six tangent circles with the same relative proportions. Such an arrangement is a special case of a *Doyle spiral*.

5 Solution: We begin by describing a strategy for Sam. As a first move, Sam crosses out 51. The remaining numbers may be divided into two groups: we will call the numbers $1,\ldots,50$ *small* and the numbers $52,\ldots,101$ *large*. Each time Nico crosses out a small number, Sam responds by crossing out the smallest large number remaining. Each time Nico crosses out a large number, Sam responds by crossing out the largest small number remaining.

(Though it will not be important in the arguments that follow, we remark that Sam's strategy can also be described as follows: always erase the median of the remaining numbers.)

There are 50 small numbers and 50 large numbers. Since Sam always responds to Nico by crossing out a number from the opposite group, after Sam's last move there will be two small numbers and two large numbers remaining. Therefore, after Nico's last move (Nico has the final move), there will be either one small and two large numbers left, or two small and one large number left.

Let the *key difference* at any point in the game refer to the difference between the largest remaining small number and the smallest remaining large number. After Sam's first move, the key difference is $52-50=2$. Note that the key difference can never decrease, and that after Sam's n^{th} move the key difference must be at least $n+1$. So after Sam's 49^{th} and last move, the key difference must be at least 50. Thus,

$$(a-b)^2+(b-c)^2+(c-a)^2 \geq 51^2+50^2+1^2 = 5102.$$

Also, note that

$$ab+bc+ca \leq 50\cdot 100+100\cdot 101+101\cdot 50 = 20150.$$

Since $20150 < 4\cdot 5102$, we have

$$ab+bc+ca < 4\big((a-b)^2+(b-c)^2+(c-a)^2\big).$$

Expanding both sides gives

$$ab+bc+ca < 8a^2+8b^2+8c^2-8ab-8bc-8ca,$$

which can be rearranged to yield the desired inequality

$$\frac{ab+bc+ca}{a^2+b^2+c^2} < \frac{8}{9}.$$